Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 7

Lecturer: Hunter Shufflebeam

Textbook Sections: 6.4, 6.5

Topics for Today:

More applications of integrals: Work and Averages of Functions

Section I: Work

Recall Newtons 2nd Law:

A force is any influence with the ability to change the motion of an object.

In modern physics, our models include four fundamental forces: the gravitational force, weak nuclear force, electromagnetic force, and strong nuclear force.



When a force $\vec{F}$ acts on an object of mass m, the object experiences an acceleration $\vec{a}$ altering its motion.

In Newtonian Mechanics, this is formulated as

$$\text{Newton's 2nd Law: } \vec{F} = m \cdot \vec{a}.$$

Take note of the units:
- m in e.g. kg
- $\vec{a}$ in e.g. m/s^2

$\rightsquigarrow \vec{F}$ in $\frac{kg \cdot m}{s^2} = J$

when a force is applied to an object over some distance, and that object experiences a change in kinetic energy, that change in energy is called the **work done on the object**.

Punchline of Section I: Let's dig into how integrals can help us calculate this fundamental quantity!

Definition: Suppose a constant force F is applied to a particle which is free to move along a straight track (the x-axis). Then the **work**, W, done on the particle by the force F is defined to be the quantity

$$W = F \cdot d \qquad \text{(units are, eg. } N \cdot m = J\text{)}$$

where d is the total distance that the particle moves.

Ex: What is the work done by gravity when a 200kg piano drops 50m?

A: Recall • $g = -9.8 m/s^2$ (acc. due to gravity at sea level)

• $\vec{F}_g = m \cdot g = -200 \cdot 9.8 = -1960$ N (- means downward!)

So: $W = F \cdot d = (-1960) \cdot (-50) = 98{,}000$ J!

Question: This is a very simple case since we assume F is constant (also, we know F_g is not constant either — what if we drop the piano from an airplane?)

In other words — can we handle variable forces?

A: Yes! We can handle very complicated paths, but doing so would require some vector calc / calc III knowledge. But for now, we can indeed handle complicated forces:

Setup: Suppose the particle moves along its linear path from $x=a$ to $x=b$, and that at each point $x \in [a,b]$, the force on the particle at point x is $f(x)$.

Provided $f(x)$ is nice (eg cts, pw. cts, etc), we can approximate the total work done on the particle over its path by

- subdividing $[a,b]$ into many short intervals I_j with length Δx
- Choosing $x_j^* \in I_j$ s.t. $f \approx f(x_j^*)$ on I_j
- Approximating the work done on I_j by $W_i \approx f(x_j^*) \cdot \Delta x$
- Adding the approximate work done on each subinterval:

$$W = \sum_{i=1}^{N} W_i \approx \sum_{i=1}^{N} f(x_j^*)\,\Delta x$$

Letting $N \to \infty$, as usual we obtain a Riemann integral. In this case, we have that

$$W = \int_a^b f(x)\,dx.$$

Examples:

(1) When P is located $(+x)$-meters from the origin, it experiences a force of $x^2 + 2x$ N. What is the work done by the force in moving P from $x=1$ to $x=3$?

A: $W = \int_1^3 f(x)\,dx = \int_1^3 (x^2+2x)\,dx = \frac{50}{3}$

(2) Recall Hooke's Law: The force required to maintain a spring stretched x m. beyond its resting length is given by (x small)

$$\vec{F} = kx$$

where k is a physical constant of the spring.

Q: If it requires 40 N to hold a spring stretched from 10 cm (rest pos) to 15 cm, how much work does it take to stretch it from 15 cm to 18 cm?

A: By Hooke's Law, $40 = k \cdot (.05) \rightsquigarrow k = 800$ N/m.

Thus,

$$W = \int_{.05}^{.08} f(x)\,dx = \int_{.05}^{.08} (800x)\,dx = 1.56 \text{ J}.$$

Section II: Averages of Functions

In the final part of todays lecture, we give another useful application of definite integrals: averaging functions:

Primer: Recall that the arithmetic average of N numbers $y_1, \dots, y_N$ is given by

$$\bar{y} = \frac{y_1 + \cdots + y_N}{N}$$

Q: Can we average the ∞-many values of a function $f: [a,b] \to \mathbb{R}$?

Let's try to approximate this average:

- Subdivide $[a,b]$ into N equally sized subintervals I_j of length $\Delta x = \frac{b-a}{N}$.
- Choose $x_j^* \in I_j$ s.t. $f \approx f(x_j^*)$ on I_j
- Approximating the true average of f on $[a,b]$ by averaging these sample values

$$\bar{F} = \text{avg of } f \text{ on } [a,b] \approx \frac{f(x_1^*) + \cdots + f(x_N^*)}{N} = \frac{1}{b-a} \sum_{i=1}^{N} f(x_i^*) \Delta x$$

- Send $N \to \infty$ to obtain

$$\bar{F} = f_{avg} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Ex: Find the avg. of $f(x) = \sin x$ on $[0, \pi]$.

$$f_{avg} = \frac{1}{\pi} \cdot \int_0^{\pi} \sin x\, dx = \frac{2}{\pi}$$

Theorem: (MVT for Integrals)

If f is cts on $[a,b]$, the there exists a $c \in [a,b]$ s.t.

$$f(c) = f_{avg} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Ex: Let $f(x) = \sin x$ on $[0, \pi]$ as above. Then $f(c) = \frac{2}{\pi}$ for at least one (but in fact two) points $c \in [0, \pi]$.